Math 3280
Review Problems for Exam II

This is not an exhaustive list of all possible types of problems. See the class notes, the textbook and homework for additional problems. You will be provided with eigenvalues and eigenvectors of certain matrices so you don't have to find them by hand.

For the following problems, consider the system

$$\frac{dx}{dt} = f(x,\, y)$$
$$\frac{dy}{dt} = g(x,\, y)$$

Find its critical points and classify its hyperbolic critical points.

1. $f(x,\, y) = x(2x - y),\ g(x,\, y) = x^2 + y - 8$
2. $f(x,\, y) = y^2 - (x^2 - 1)(x - 2),\ g(x,\, y) = x - y - 2$
3. $f(x,\, y) = x^2 - y,\ g(x,\, y) = xy + y^2$
4. $f(x,\, y) = x(3 - x - y),\ g(x,\, y) = y(2 - x/2 - y)$
5. $f(x,\, y) = x(3 - x + y),\ g(x,\, y) = y(2 + x - 2y)$
6. $f(x,\, y) = 1 - x^2 - y^2,\ g(x,\, y) = x - y$

Draw its phase portraits by considering the following. (a) Find critical points and classify them, if possible (b) Both x and y nullclines. (c) Direction vectors in each of the regions the phase plane is divided into by nullclines. (d) Local behavior of trajectories near critical points. (e). Isoclines, invariant lines or curves.

7. $f(x,\, y) = y,\ g(x,\, y) = -x + (x^2 - 1)y$
8. $f(x,\, y) = x - xy - 1,\ g(x,\, y) = y + xy$
9. $f(x,\, y) = x(2x - y),\ g(x,\, y) = x^2 + y - 8$
10. $f(x,\, y) = x(3 - x - y),\ g(x,\, y) = y(2 - x/2 - y)$. Graph the phase portrait only in the first quadrant and discuss the long-term behavior if $x(0) > 0$ and $y(0) > 0$.
11. $f(x,\, y) = x(3 - x + y),\ g(x,\, y) = y(2 + x - 2y)$. Graph the phase portrait only in the first quadrant and discuss the long-term behavior if $x(0) > 0$ and $y(0) > 0$.
12. $f(x,\, y) = 1 - x^2 - y^2,\ g(x,\, y) = x - y$

(Partial) Answers to the Review Problems for Exam II Continued

17. All critical points are on straight-line solutions.
18. The only critical point not on a straight-line solution is a saddle point.
19. The limit cycle can only be in the first quadrant and in that region $\nabla \cdot \left(\frac{1}{xy} \begin{bmatrix} f \\ g \end{bmatrix} \right) \neq 0$.
20. (a) Critical points not on a straight-line solution are saddles. (b) Let $\rho = 1$.
21. $r = 1$ semistable; $r = 2$ unstable; $r = 3$ stable
22. See the lecture notes.
23. Just follow the required steps.
24. Show that $R = \{(r, \theta) : 1 \leq r \leq 3\}$ satisfies the Poincarè-Bendixon theorem.
25. Let $\rho = 1$ and show $\nabla \cdot (\rho \vec{F}) < 0$ in annulus A by converting it to polar coordinates.

Now, x and y represent two species. Describe their interaction. Can either species survive in the absence of the other? Hint: Consider the system when $x(t) = 0$ or $y(t) = 0$ for all t.

13. $f(x, y) = x(3 - x - y)$, $g(x, y) = y(2 - x/2 - y)$.
14. $f(x, y) = x(3 - x + y)$, $g(x, y) = y(2 + x - 2y)$.
15. $f(x, y) = x(3x - x^2 + y)$, $g(x, y) = y(-2 + x)$.
16. $f(x, y) = x^2(3 - x - y)$, $g(x, y) = y(-2 + x)$.

Show that the following systems cannot have a limit cycle.

17. $f(x, y) = x(2 - x + y)$, $g(x, y) = y(1 - x - y)$.
18. $f(x, y) = x(2 - x - y)$, $g(x, y) = y(1 - x - 2y)$.
19. $f(x, y) = x(3 - x - y)$, $g(x, y) = y(2 - x/2 - y)$.
20. Show that the system with $f(x, y) = x(2y + 1)$, $g(x, y) = x^2 - y^2$ doesn't have a limit cycle two different ways. (a) Analysis of the critical points. (b) Bendixon-Dulac Criterion.
21. Consider the system $\frac{dr}{dt} = r(r-1)^2(2-r)(r-3)$, $\frac{d\theta}{dt} = 1$ given in polar coordinates. Find and classify its limit cycles.
22. Consider the system $f(x, y) = -y + x(1 - x^2 - y^2)$, $g(x, y) = x + y(1 - x^2 - y^2)$. Convert this system to polar coordinates. Find its limit cycle and draw the phase portrait.
23. Consider the system $f(x, y) = -x - y + xy^2 + x^3$, $g(x, y) = x - y + x^2y + y^3$. Convert this system to polar coordinates. Analyze the de-coupled system in polar coordinates. Show $r = 1$ is an unstable limit cycle and draw the phase portrait of this system by hand.
24. Consider the system $\frac{dr}{dt} = r(6 + 2\sin\theta - r^2)$, $\frac{d\theta}{dt} = -3 + \cos\theta$ given in polar coordinates. The only critical point of this system is the origin. Prove that this system has a limit cycle.
25. Consider the system $\vec{x}\,'(t) = \vec{F}(\vec{x})$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{F}(\vec{x}) = \begin{bmatrix} -y + x(3 - 2x^2 - 3y^2) \\ x + y(3 - 2x^2 - 3y^2) \end{bmatrix}$. Prove that this system has at most one limit cycle entirely contained in the polar coordinates annulus $A = \{(r, \theta) : 1 \le r \le 2\}$. Notes: Do not convert the system to polar coordinates. Do not prove existence of a limit cycle.

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(Partial) Answers to the Review Problems for Exam II

1. $(-4, -8)$ spiral sink; $(0, 8)$ saddle; $(2, 4)$ spiral source
2. (2, 0) sink
3. (0, 0) Nonhyperbolic; $(-1, 1)$ saddle
4. (0, 0) source; (0, 2) saddle; (3, 0) saddle; (2, 1) sink
5. (0, 0) source; (0, 1) saddle; (3, 0) saddle; (8, 5) sink
6. $(-1/\sqrt{2}, -1/\sqrt{2})$ saddle; $(1/\sqrt{2}, 1/\sqrt{2})$ spiral sink
7. The only critical point (0, 0) is a spiral sink. Vertical tangent lines on the x-axis.
8. The critical point (1, 0) is a source, while $(-1, 2)$ is a saddle point with its stable manifold tangent to $y - 2 = -(x + 1)$ and it unstable manifold tangent to $y - 2 = 2(x + 1)$. The x-axis is a straight-line solution. For the hyperbola $y = (x - 1)/x$, between its two curves, x is decreasing while to the left and right of them x is increasing. To the right of $x = -1$, y is increasing if $y > 0$ and decreasing if $y < 0$ and switched to the left of $x = -1$.
9. The y-axis is the unstable manifold of (0, 8).
10. The x- and y-axes are the stable manifolds of critical points (3, 0) and (0, 2), respectively, while their unstable manifolds, in the first quadrant will converge to the critical point (2, 1).
11. The x- and y-axes are the stable manifolds of critical points (3, 0) and (0, 1), respectively, while their unstable manifolds, in the first quadrant will converge to the critical point (8, 5).
12. Consider regions inside and outside the circle $x^2 + y^2 = 1$ and above and below the line $y = x$ to draw the vectors.
13. Competition model. Either species can survive in the absence of the other.
14. Symbiosis model. Either species can survive in the absence of the other.
15. Symbiosis model. Only x can survive in the absence of y.
16. x represents the prey and y is the predator. Only the prey can survive in the absence of the predator.